TRUNCATED POWER-SERIES

by

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Truncated power-series.

1. Introduction.

More than once already work has been done concerning theorems on truncated power-series, i.e. those polynomials that are obtained by truncating a complete series expansion of some function or other after a certain term.

In the first place there is the work of O. Blumenthal [1], who considers the truncated series of the function $(1+x)^{\delta}$ in his paper: "La geométrie des polynômes binomiaux".

Further there are the several asymptotic expansions on series of the exponential type, the first of which we owe to a conjecture of Ramanujan.[2,3,4,5,6].

Finally there are a number of papers that deal with parts of power series which remain limited within a certain area, among others by W. Rogosinski and G. Szegö [7].

Now it is the intention to go into the matter in a similar way as Blumenthal did; no longer, however, for the binomial polynomials, but for those polynomials that are obtained by truncating the series of the exponential type. In doing this, one or two things of the asymptotic expansions will be required, which expansions are recorded in the references (2) up to and including (6), but there will be no need to use the third kind of papers.

The list of authors that have been quoted is not complete, and what has been treated in the following passage remains rather elementary.

2. Definitions.

First we introduce the following symbols, n is a natural number

$$E_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} , \tag{2.1}$$

$$C_{2n}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} \tag{2.2}$$

and $$S_{2n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \tag{2.3}$$

The following relations are well-known:

$$\lim_{n\to\infty} \begin{Bmatrix} E_n(x) & = \exp x. \\ C_{2n}(x) & = \cos x. \\ S_{2n+1}(x) & = \sin x. \end{Bmatrix} \tag{2.4}$$

3. Some properties of $E_n(x)$.

To begin with we examine (2.1) a little closer. There are two types of polynomials and these types are most clearly characterised by their behaviour for negative x.

Utilizing the remainder representation of Lagrange the following relation is easily found:

$$\exp x - E_n(x) = \int_0^x \exp u \frac{(x-u)^n}{n!} du . \tag{3.1}$$

So for $x > 0$ it holds good universally that

$$\exp x > E_n(x), \tag{3.2}$$

in which for every n an x can be chosen in such a way, that the difference of the two functions exceeds any positive value, and with fixed x n can be chosen so, that the difference of the two functions may take any small value.

For $x > 0$ and $n > m$ of all values it holds good that

$$E_n(x) > E_m(x), \tag{3.3}$$

as $E_n(x)$ has more terms (all of them being positive) than $E_m(x)$. So the integral in (3.1) is a function of x, which goes down monotonously

For $x = 0$ is $E_n(0) = 1$ for every n.

For $x < 0$ assume that $x = -y$.

$$\exp(-y) - E_n(-y) = (-)^{n+1} \int_0^y \exp(-u) \frac{(y-u)^n}{n!} du. \tag{3.4}$$

$$\left.\begin{array}{ll} \text{If n is even, then} & E_n(-y) > \exp(-y), \\ \text{If n is odd , then} & E_n(-y) < \exp(-y), \end{array}\right\} \tag{3.5}$$

and again, with fixed n, the difference of the two functions can be made to take on any large value, whereas with fixed y the difference may take on any small value.

Therefore the functions with even index-number x cannot possibly have a real zero.

In what follows "zero" will always signify "real zero". On account of

$$\frac{d\, E_n(x)}{dx} = E_{n-1}(x) \tag{3.6}$$

the functions with an even index-number cannot but go up monotonously and so these functions have only one zero.

A polynomial from the even class has a minimum in the place where its predecessor from the odd class has a zero, and these curves have no points of inflection.

The odd class does have a zero and a point of inflexion but no maximum or minimum.

The place next to the zeros will be further stated in section 5.

It is easily seen that

$$E_n(-n) - E_{n-2}(-n) = 0 \tag{3.7}$$

i.e. $E_n(x)$ and $E_{n-2}(x)$ meet in the points $x = 0$ and $x = -n$.

For the time being we consider the polynomials from the even class. The properties that are derived below can easily be transferred to the polynomials of the odd class.

So $E_{2n}(x)$ and $E_{2n-2}(x)$ meet in the points $x = 0$ and $x = -2n$ within the interval $-2n < x < 0$, $E_{2n}(x)$ is smaller than E_{2n-2}, outside the interval it is larger.

$E_{2n-2}(x)$ and $E_{2n-4}(x)$ meet in the points $x = 0$ and $x = -(2n-2)$; within the interval $-(2n-2) < x < 0$, $E_{2n-2}(x)$ is smaller than $E_{2n-4}(x)$, outside the interval it is larger.

So within the interval $-(2n-2) \leqslant x < 0$, $E_{2n-4}(x)$ is larger than $E_{2n}(x)$, for $x > 0$ $E_{2n}(x)$ is larger than $E_{2n-4}(x)$ and for $x < -2n$, $E_{2n}(x)$ is larger than $E_{2n-4}(x)$.

So beside in $x = 0$, $E_{2n-4}(x)$ and $E_{2n}(x)$ have only intersections within the interval $-2n < x < -(2n-2)$. There is sure to be one intersection and there are three at most. Suppose there are three intersections. Then it follows from the theorem of Rolle, that $E_{2n-1}(x)$ and $E_{2n-5}(x)$ too must have three intersections within the interval $-(2n-1) < x < -(2n-5)$. If we go on like this with Rolle we arrive at the conclusion that $E_4(x)$ and $E_0(x) = 1$ must have four intersections. This is certainly not true, as $E_4(x)$ has only one minimum. Therefore $E_{2n-4}(x)$ and $E_{2n}(x)$ have two intersections. In a similar manner it is seen that $E_{2n}(x)$ and $E_{2m}(x)$, with $m \neq n$, have only two intersections, and that for n approaching infinity the absciss of one intersection moves towards $-\infty$, the other remaining $x = 0$ all the time.

4. Some properties of $C_{2n}(x)$ and $S_{2n+1}(x)$.

Without restriction nothing but $x > 0$ can be considered now, this in connection with the odd/ even character of the polynomials.

As in section 3 one will find:

$$\cos x - C_{2n}(x) = (-1)^{n+1} \int_0^x \sin y \frac{(x-y)^{2n}}{(2n)!} dy \qquad (4.1)$$

and

$$\sin x - S_{2n+1}(x) = (-1)^{n+1} \int_0^x \sin y \frac{(x-y)^{2n+1}}{(2n+1)!} dy \qquad (4,2)$$

For even n it holds good that:

$$\left.\begin{matrix} C_{2n}(x) \\ S_{2n+1}(x) \end{matrix}\right\} \text{ larger than } \left\{\begin{matrix} \cos x \\ \sin x \end{matrix}\right. ,$$

and for odd n:

$$\left.\begin{matrix} C_{2n}(x) \\ S_{2n+1}(x) \end{matrix}\right\} \text{ smaller than } \left\{\begin{matrix} \cos x \\ \sin x \end{matrix}\right. .$$

Further one has now:

$$\frac{d}{dx} C_{2n}(x) = S_{2n-1}(x) \qquad (4.3)$$

and

$$\frac{d}{dx} S_{2n+1}(x) = - C_{2n}(x).$$

So $C_{2n}(x)$ has extremities in the zeros of $S_{2n-1}(x)$ and points of inflection in the zeros of $C_{2n-2}(x)$.

The question how many zeros $C_{2n}(x)$ respectively $S_{2n+1}(x)$ have, will be dealt with later on.

The intersections of the curves $C_{2n}(x)$ and $C_{2n-4}(x)$ are easy to compute, namely $x = 0$ and $x = \pm\sqrt{2n(2n-1)}$. So for $x > 0$ $C_{2n}(x)$ and $C_{2n-4}(x)$ have only one intersection. In a similar way as for the polynomials $E_n(x)$, it can be proved that $C_{2n}(x)$ only meets each $C_{2m}(x)$ in one point, for $x > 0$ and that, if n and m differ two and four. The limit for n approaching ∞ of this intersection is again infinite.

Generally $C_{2m}(x)$ and $C_{2n}(x)$ have more intersections. For if we choose $m = 0$ for instance, then is $C_0(x) = 1$ and for a large n there are also a large number of intersections. The number of intersections of $C_{2n}(x)$ and $C_0(x)$ is, very roughly approximated, equal to the number of zeros of $C_{2n}(x)$. It is clear that the number of intersections is larger or equal to the number of zeros. For this also see section 6.

For even n the values of $C_{2n}(x)$ are near those of x, which are smaller than $k\pi$ (k entirely > 0), whereas these extremities for odd n occur with values of x that are larger than $k\pi$. The approach of these extremities towards $k\pi$ is monotonous.

Similar properties also hold good for the $S_{2n+1}(x)$.

5. Asymptotic approximation of the zero of $E_n(x)$.

The problem is to determine for large n the behaviour of the function $\xi = \xi(n)$ which satisfies the relation $E_n(-\xi) = 0$ and $\xi \neq 0$.

In a forthcoming report of the Computation Department Mathematical Centre it is shown that:

$$E_n(-y) = (-1)^n \frac{y^n}{n!} I(y,n) + \exp(-y) \tag{5.1}$$

in which

$$I(y,n) = \int_0^y e^{-u}\left(1 - \frac{u}{y}\right)^n du \tag{5.2}$$

For I(y,n) the following asymptotic expansion holds good:

$$I(y,n) \sim \frac{y}{n+y} - \frac{n\,y}{(n+y)^3} + \dots \tag{5.3}$$

For y there are certain restrictions, but for $y > 0$ the expansion that is applied here is certainly allowed.

For $y = \xi$ the right hand term of (5.1) equals rought provided that n is odd.

If we introduce further that

$$a = \frac{\xi}{n} \tag{5.4}$$

and if for a the following asymptotic series is substituted in (5.1):

$$a = a_0 + \frac{b_1}{n}\log n + \frac{a_1}{n} + \frac{c_1}{n^2}\log^2 n + \dots \tag{5.5}$$

it is possible to compute a_n, b_n, c_n etc.

Here only a_0, a_1 and b_1 will be determined.

$$\frac{a^n n^n}{n!}\left(\frac{a}{1+a} - \frac{a}{(1+a)^3 n} + \dots\right) = \exp(-an).$$

According to Stirling is

$$n! = \sqrt{2\pi}\; e^{-n}\, n^{n+\frac{1}{2}} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} \dots\right)$$

and therefore is

$$\exp(-a) = e.a \sqrt[n]{\frac{T}{N}} \tag{5.6}$$

in which $T = \frac{a}{1+a} - \frac{a}{(1+a)^3 n} + \dots$

and $N = \sqrt{2\pi n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} \dots\right)$

It is easy to compute that:

$$\sqrt[n]{\frac{T}{N}} = \sqrt[n]{\frac{a_0}{(1+a_0)\sqrt{2\pi n}}}\left\{1 + 0\left(\frac{\log n}{n^2}\right)\right\}$$

So a_0 can be found from

$$\exp(-a_0 - 1) = a_0$$

$$\text{or } a_0 = 0.278464 \tag{5.7}$$

Further

$$b_1 = \frac{a_0}{2(1+a_0)} = 0.108906 \tag{5.8}$$

$$\text{and} \quad a_1 = \frac{a_0}{2(1+a_0)} \log \frac{a_0^2}{(1+a_0)^2 2\pi} = -0.532124 \tag{5.9}$$

6. The number of zeros of $C_{2n}(x)$ and $S_{2n+1}(x)$.

In the table below the order of the C or S-polynomial is stated, with the number of real zeros of the polynomial in the column next to it.

C_0	0	S_5	1	C_{10}	2
S_1	1	C_6	2	S_{11}	3
C_2	2	S_7	3	C_{12}	4
S_3	3	C_8	4	S_{13}	5
C_4	4	S_9	5	C_{14}	6

This table can be easily verified with the help of very simple means. For if a polynomial has a complex zero ξ , then also its complex conjugated ξ^* is a zero.

But $-\xi$ instead of ξ must also be a zero, on account of the odd/even properties.

So the complex zeros always appear in fours. This is the proof of the first row of the table.

Now $S_5(x) = x\ (1 - \frac{x^2}{6} + \frac{x^4}{120})$ is a polynomial which for $x > 0$ remains larger than sin x.

The extremities of $(1 - \frac{x^2}{6} + \frac{x^4}{120})$ lie near $x = \sqrt{10}$, but the value of the minimum is still positive. Therefore $S_5(x)$ has one real zero and four complex ones. If $C_6(x)$ should have six real zeros (there are either two or six), then, according to Rolle, $S_5(x)$ should have at least five zeros.

Thus we get at $C_8(x)$. But for $x < \sqrt{56}$ $C_4(x)$ is larger than $C_8(x)$, which , in its turn is larger than cos x. So $C_8(x)$ has at least as many zeros as $C_4(x)$ has, a thing which also holds good universally for any set of polynomials that differ 4 in order.

In order to show that $S_9(x)$ has five real zeros, the only thing to do is to compute $S_9(4) = -0.663$. And a few more computations yield the last row of the table.

There is a certain regularity, but from what follows it will appear that once this regularity is disturbed.

From the formulas (5.1), (5.2) and (5.3) it appears that

$$C_{2n}(x) = \sum_{k=0}^{n} (-)^k \frac{x^{2k}}{(2k)!} = \cos x + (-1)^{n+1} \frac{x^{2n}}{(2n)!} \left(\frac{x^2}{4n^2+x^2} + \dots\right) \tag{6.1}$$

The absolute value of cos x is bounded by 1. The term $\frac{x^{2n}}{(2n)!} \cdot \frac{x^2}{4n^2+x^2}$ may exceed this value and if we want to make it possible for $C_{2n}(x)$ to have a zero, the value of x will have to be chosen so, that this term is not larger than 1. The extremities of cos x are near $x = r\pi$, and therefore in these points the term mentioned above must not exceed the value 1.

$$\frac{(r\pi)^{2n}}{2n!} \; \frac{(r\pi)^2}{4n^2+(r\pi)^2} \approx 1 .$$

If it is assumed that

$$\frac{r}{n} = q = q_0 + \frac{q_1}{n} + \frac{q_2}{n^2} + \dots + \frac{q_1'}{n} \log n + \dots \tag{6.2}$$

then it appears that:

$$\frac{qe\pi}{2} = \sqrt[4n]{\left(\frac{4+q^2\pi^2}{q^2\pi^2}\right)^2 4\pi n \left[1 - 0\left(\frac{1}{n}\right)\right]} .$$

So $q_0 = \frac{2}{\pi e} = 0.23419933$ (6.3)

and $q_1' = \frac{e\pi}{2} = 4.2698670$ (6.3)

So for large n the number of zeros approaches $\left[n.q_0 + q' \log n\right]$. As $q_0 > \frac{1}{5}$, it appears that the table, which is given at the head of this section, cannot be continued without more ado. It is very difficult to determine the exact number of zeros. As was already noted in [6] , it is not possible to evaluate cos n with the help of an asymptotic representation. Here cos x will have to be evaluated with other expedients, and then, with the help of (6.1), it can be verified whether $C_{2n}(x)$ can possibly have a zero in x.

To guess cos x analytically, in which x is obtained by multiplying the asymptotic series by $n\pi$, is not a pleasant job.

Numerically for every n surely a q_1 can be found by means of iteration and then it is also possible to determine the number of zeros. For every n a certain amount of computations have to be performed.

The number of intersections of $C_{2n}(x)$ and $C_0(x)$ has also an asymptotic representation in the form of (6.2), in which q_0 and q_1' do not change.

Generally speaking, $C_{2n}(x)$ and $C_{2m}(x)$ have no more intersections than $C_0(x)$ and $C_{2n-2m}(x).(n > m)$. This is to be derived again via the theorem of Rolle.

For the $S_{2n+1}(x)$-polynomials a similar treatise can be held and then it appears that (6.2) and (6.3) can be maintained.

7. A next step.

Functions which resemble closely the exponential functions are the Bessel-Functions.

In order to examine the truncated power-series of these functions, (7.1) is introduced:

$$I_{N,k}(x) = \sum_{h=0}^{N} \frac{x^h}{h!(h+k)!} \qquad \left.\begin{matrix} k = 0,1,2\ldots \\ N = 0,1,2,\ldots \end{matrix}\right\} \tag{7.1}$$

This way of defining gives an advantage: Both Besselian functions are dealt with at the same time, for

$$\lim_{\substack{N\to\infty \\ x \geqslant 0}} I_{N,k}(x) = I_k(2\sqrt{x})/(\sqrt{x})^k \tag{7.2}$$

$$\lim_{\substack{N\to\infty \\ x \geqslant 0}} I_{N,k}(-x) = J_k(2\sqrt{x})/(\sqrt{x})^k \tag{7.2}$$

By means of differentiation one has:

$$\frac{d}{dx} I_{N,k}(x) = I_{N-1,k+1}(x) \tag{7.3}$$

Similar contemplations as with exp x, for $x \geqslant 0$, give for $N > M$

$$1 \leqslant I_{N,k}(x) \leqslant I_{N,k}(x) \leqslant I_k(2\sqrt{x})/(\sqrt{x})^k \,, \tag{7,4}$$

All signs of equality are dropped in (7.4) if $x > 0$. Thus it has been shown that with fixed k the functions (7.1) are monotonous.

Utilizing the well-known theorem of Hurwitz it follows from the last equality (7.2) for $x < 0$ that the functions $I_{N,k}(x)$ can have any dictated number of zeros, provided that N is sufficiently large.

And here too the problem arises to determine the number of zeros as a functions of N, with k already choosen. That this will not be simple is clear, after what has been seen in the cosine and sine-case. Add to this that $J_k(2\sqrt{x})$ is no longer a periodical function and that the asymptotic expansions mentioned in the sections 5 and 6 have up till now not yet been proved for the Bessel Functions.

It is easy to see that $I_{1,k}(x)$ has one and only one zero $[x = -(k+1)]$. $I_{2,k}(x)$ have no zeros for $k > 0$, but a minimum, and a minimum-double zero at $x = -2$. According to (7.3) $I_{3,k}(x)$ are again functions of x that go up monotonously, and that have only one zero.

If $R_{N,k}(x) = I_{\infty,k}(x) - I_{N,k}(x)$,

$I_{\infty,k}(x) = \lim_{N\to\infty} I_{N,k}(x)$,

and $x \geqslant 0$

than it is to be found from the remainder representation of Lagrange that:

$$R_{N,k}(x) = \int_0^x I_{\infty,k}^{(N+1)}(y) \frac{(x-y)^N}{N!}\,dy$$

$$= \int_0^x I_{\infty,k+N+1}(y) \frac{(x-y)^N}{N!}\,dy, \qquad (7.5)$$

and that: $$R_{N,k}(-x) = (-1)^{N+1} \int_0^x I_{\infty,k+N+1}(-u) \frac{(x-u)^N}{N!}\,du \;. \qquad (7.6)$$

The conclusion, which follows directly from (7.5), that this remainder is always positive, has already been derived in another manner. In connection with (7.2), (7.6) becomes

$$R_{N,k}(-x) = \frac{(-1)^{N+1}}{N!} \int_0^x J_{k+N+1}(2\sqrt{u})\,(x-u)^N\,u^{-\frac{k+N+1}{2}}\,du \qquad (7.7)$$

If now the transformation

$$u = \frac{t^2}{4}$$

is carried out, (7.7) is tranferred into

$$R_{N,k}(-x) = \frac{(-1)^{N+1}2^{k+N}}{N!} \int_0^{2\sqrt{x}} J_{k+N+1}(t)\,\frac{(x-\frac{t^2}{4})}{t^{k+N}}\,dt \;. \qquad (7.8)$$

The object is to show that the integral in (7,8) is positive for all values $x > 0$.

If for convenience it is assumed that $\nu = k+N+1$ and if for $J_\nu(t)$ the integral representation (7) is used, which is mentioned on page 48, Theory of Bessel Functions by G.W. Watson, then is:

$$J_\nu(t)=\frac{(2\nu-1)(\frac{t}{2})^{\nu-1}}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^{\frac{\pi}{2}} \sin(t\cos\Theta)\sin^{2\nu-2}\Theta\cos\Theta d\Theta \qquad (7.9)$$

in which $\mathrm{Re}\,\nu > \frac{1}{2}$.

The factor $\frac{(2\nu-1)2^{-\nu+1}}{\Gamma(\nu+\frac{1}{2})\Gamma(\frac{1}{2})}$ is positive then and can be put before both integrations.

It only remains to proof that

$$\int_0^{2\sqrt{x}}dt\int_0^{\frac{\pi}{2}}d\Theta\,(x-\frac{t^2}{4})^N\sin(t\cos\Theta)\sin^{2\nu-2}\Theta\cos\Theta \geqslant 0 \qquad (7.10)$$

for $x \geqslant 0$.

Within the interval $0 \leqslant t \leqslant 2\sqrt{x}$, $(x-\frac{t^2}{4})$ is a function that goes down monotonously. Therefore

$$\int_0^{2\sqrt{x}}(x-\frac{t^2}{4})^N\sin(t\cos\Theta)dt \geqslant 0 \qquad (7.11)$$

for any $0 \leqslant \Theta \leqslant \frac{\pi}{2}$.

(7.10) is easy to be derived now.

By confining oneself to values of $x > 0$ it can even be shown that the sign of equality in (7.10) can be dropped.

Now the question concerning the number of intersections of $I_{N,k}(x)$ and $I_{M,k}(x)$ arises again. Only for those values of N and M that differ an even number is beside $x = 0$ another intersection possible. However, there can also be more intersections.

From the fact that $I_{4,k}(x)$ can have only one minimum, it can be derived, as in the cosine-case, that every two functions with $N-M=4$ can have only two intersections. $x = 0$ is always one of them, the other is near a negative value of x and for limit $N\to\infty$ this value approaches $-\infty$.

The intersection of $I_{N,k}(x)$ and $I_{N-2,k}(x)$ is near $x = -N(N+k)$. Hence it follows that, when $I_{N-2,k}(x)$ is supposed to have ν zeros, $I_{N-2,k}(x)$ has at least ν zeros too. The cosine, respectively sine-case is different in this respect that if the function has a complex zero, also the complex conjugated is a zero, but not the one that is reflected in the origin.

8. Conclusion.

It is clear that the treatises held here have one parameter lees than the work of O. Blumenthal. For he has beside n also γ at his disposal, $(1+x)^{\gamma}$. But the asymptotic determination of the zero in the case of the $E_n(x)$-polynomials and the asymptotic determination of the number of zeros in the case of the $C_{2n}(x)$ or $S_{2n+1}(x)$-polynomials, has been added.

This attempt was successful via asymptotic expansions which for the computation of the functions concerned, exp(-x), cos x and sin x have no significance whatever.

The polynomials could also be regarded in the complex plane, in which case one does not confine oneself to real values of x.

Finally, an explanation (which is not yet completed) on polynomials which are obtained by truncating the Besselian Functions has been added.

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References.

1. O. Blumenthal, La geométrie des polynômes binomiaux. C.R. du Congres Liège 1939, pages 69-74.
2. Collected Papers of Srinivasa Ramanujan, (1927), XXVI, VII. Theorems on approximate integration and summation of series.
3. G. Szegö, Ueber einige von Ramanujan gestellte Aufgaben. J.London Math.Soc., (3) 31 (1928), 225-232.
4. G.N. Watson, Theorems stated by Ramanujan. Proc.London Math.Soc., (2) 29 (1928), 293-308.
5. E.T. Copson, An approximation connected with e^{-x}. Proc.Edinb.Math.Soc., (2) 3 (1933), 201-206.
6. J. Berghuis, An approximation connected with cos n and sin n. R 72, Computation Department Mathematical Centre.
7. W. Rogosinski and G. Szegö, Ueber die Abschnitte von Potenzreihen, die in einem Kreise beschränkt bleiben. Math.Zeitschrift (28) 1 (1928), (73-94).
8. Polya und Szegö, Aufgaben und Lehrsätze, Vol.2, page 48.
9. G.N. Watson, Theory of Bessel Functions, page 48.